SIMPLE LOOPS ON 2-BRIDGE SPHERES IN 2-BRIDGE LINK COMPLEMENTS

DONGHI LEE AND MAKOTO SAKUMA

Abstract. The purpose of this note is to announce complete answers to the following questions. (1) For an essential simple loop on a 2-bridge sphere in a 2-bridge link complement, when is it null-homotopic in the link complement? (2) For two distinct essential simple loops on a 2-bridge sphere in a 2-bridge link complement, when are they homotopic in the link complement? We also announce applications of these results to character varieties and McShane's identity.

1. Introduction

Let K be a knot or a link in S^3 and S a punctured sphere in the complement $S^3 - K$ obtained from a bridge sphere of K. Then the following natural question arises.

Question 1.1. (1) *Which essential simple loops on S are null-homotopic in $S^3 - K$?*

(2) *For two distinct essential simple loops on S, when are they homotopic in $S^3 - K$?*

A refined version of the first question for 2-bridge spheres of 2-bridge links was proposed in the second author's joint work with Ohtsuki and Riley [20, Question 9.1(2)], in relation with epimorphisms between 2-bridge links. It may be regarded as a special variation of a question raised by Minsky [9, Question 5.4] on essential simple loops on Heegaard surfaces of 3-manifolds.

The purpose of this note is to announce a complete answer to Question 1.1 for 2-bridge spheres of 2-bridge links established by the series of papers [11, 12, 13, 14] and to explain its application to the study of character varieties and McShane's identity [15].

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The key tool for solving the question is small cancellation theory, applied to two-generator and one-relator presentations of 2-bridge link groups. We note that it has been proved by Weinbaum [32] and Appel and Schupp [5] that the word and conjugacy problems for prime alternating link groups are solvable, by using small cancellation theory (see also [10] and references in it). Moreover, it was shown by Sela [24] and Préaux [21] that the word and conjugacy problems for any link group are solvable. A characteristic feature of our work is that it gives complete answers to special (but also natural) word and conjugacy problems for the link groups of 2-bridge links, which form a special (but also important) family of prime alternating links. (See [1, 4] for the role of 2-bridge links in Kleinian group theory.)

This note is organized as follows. In Sections 2, 3 and 4, we describe the main results, applications to character varieties and McShane's identity. The remaining sections are devoted to explanation of the idea of the proof of the main results. In Section 5, we describe the two-generator and one-relator presentation of the 2-bridge link group to which small cancellation theory is applied, and give a natural decomposition of the relator, which plays a key role in the proof. In Section 6, we introduce a certain finite sequence associated with the relator and state its key properties. In Section 7, we recall small cancellation theory and present a characterization of the "pieces" of the symmetrized subset arising from the relator. In Sections 8 and 9, we describe outlines of the proofs of the main results.

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2. Main results

For a rational number $r \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, let $K(r)$ be the 2-bridge link of slope r, which is defined as the sum $(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$ of rational tangles of slope ∞ and r (see Figure 1). The common boundary $\partial(B^3, t(\infty)) = \partial(B^3, t(r))$ of the rational tangles is identified with the *Conway sphere* $(\boldsymbol{S}^2, \boldsymbol{P}) := (\mathbb{R}^2, \mathbb{Z}^2)/H$, where H is the group of isometries of the Euclidean plane $\mathbb{R}^2$ generated by the π-rotations around the points in the lattice $\mathbb{Z}^2$. Let $\boldsymbol{S}$ be the 4-punctured sphere $\boldsymbol{S}^2 - \boldsymbol{P}$ in the link complement $S^3 - K(r)$. Any essential simple loop in $\boldsymbol{S}$, up to isotopy, is obtained as the image of a line of slope $s \in \hat{\mathbb{Q}}$ in $\mathbb{R}^2 - \mathbb{Z}^2$ by the covering projection onto $\boldsymbol{S}$. The (unoriented) essential simple loop in $\boldsymbol{S}$ so obtained is denoted by α_s. We

also denote by α_s the conjugacy class of an element of $\pi_1(\boldsymbol{S})$ represented by (a suitably oriented) α_s. Then the *link group* $G(K(r)) := \pi_1(S^3 - K(r))$ is identified with $\pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle$, where $\langle\langle \cdot \rangle\rangle$ denotes the normal closure.

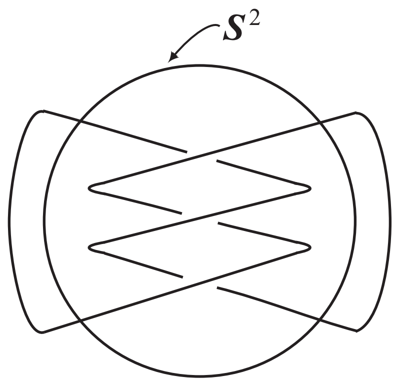

FIGURE 1. $(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$ with $r = 1/3$. Here $(B^3, t(r))$ and $(B^3, t(\infty))$, respectively, are the inside and the outside of the bridge sphere $\boldsymbol{S}^2$.

Let $\mathcal{D}$ be the *Farey tessellation*, whose ideal vertex set is identified with $\hat{\mathbb{Q}}$. For each $r \in \hat{\mathbb{Q}}$, let Γ_r be the group of automorphisms of $\mathcal{D}$ generated by reflections in the edges of $\mathcal{D}$ with an endpoint r, and let $\hat{\Gamma}_r$ be the group generated by Γ_r and Γ_∞. Then the region, R, bounded by a pair of Farey edges with an endpoint ∞ and a pair of Farey edges with an endpoint r forms a fundamental domain of the action of $\hat{\Gamma}_r$ on $\mathbb{H}^2$ (see Figure 2). Let I_1 and I_2 be the closed intervals in $\hat{\mathbb{R}}$ obtained as the intersection with $\hat{\mathbb{R}}$ of the closure of R. Suppose that r is a rational number with $0 < r < 1$. (We may always assume this except when we treat the trivial knot and the trivial 2-component link.) Write

$$r = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \ddots + \cfrac{1}{m_k}}} =: [m_1, m_2, \dots, m_k],$$

where $k \geq 1$, $(m_1, \ldots, m_k) \in (\mathbb{Z}_+)^k$, and $m_k \geq 2$. Then the above intervals are given by $I_1 = [0, r_1]$ and $I_2 = [r_2, 1]$, where

$$r_1 = \begin{cases} [m_1, m_2, \ldots, m_{k-1}] & \text{if } k \text{ is odd,} \\ [m_1, m_2, \ldots, m_{k-1}, m_k - 1] & \text{if } k \text{ is even,} \end{cases}$$

$$r_2 = \begin{cases} [m_1, m_2, \ldots, m_{k-1}, m_k - 1] & \text{if } k \text{ is odd,} \\ [m_1, m_2, \ldots, m_{k-1}] & \text{if } k \text{ is even.} \end{cases}$$

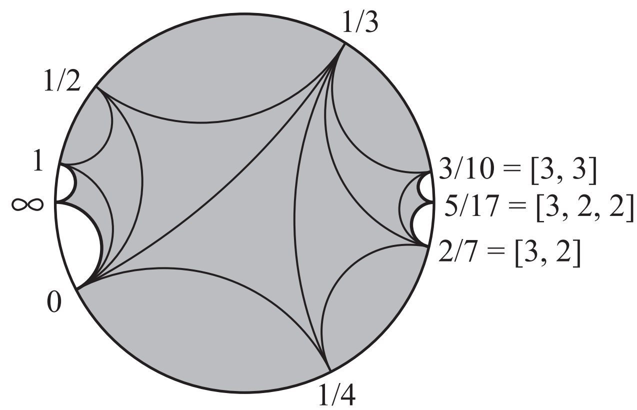

FIGURE 2. A fundamental domain of $\hat{\Gamma}_r$ in the Farey tessellation (the shaded domain) for $r = 5/17 = [3, 2, 2]$.

We recall the following fact ([20, Proposition 4.6 and Corollary 4.7] and [11, Lemma 7.1]) which describes the role of $\hat{\Gamma}_r$ in the study of 2-bridge link groups.

Proposition 2.1. (1) *If two elements s and s' of $\hat{\mathbb{Q}}$ belong to the same orbit $\hat{\Gamma}_r$-orbit, then the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$.*

(2) *For any $s \in \hat{\mathbb{Q}}$, there is a unique rational number $s_0 \in I_1 \cup I_2 \cup \{\infty, r\}$ such that s is contained in the $\hat{\Gamma}_r$-orbit of s_0. In particular, α_s is homotopic to α_{s_0} in $S^3 - K(r)$. Thus if $s_0 \in \{\infty, r\}$, then α_s is null-homotopic in $S^3 - K(r)$.*

Thus the following question naturally arises (see [20, Question 9.1(2)]).

Question 2.2. (1) *Does the converse to Proposition 2.1(2) hold? Namely, is it true that α_s is null-homotopic in $S^3 - K(r)$ if and only if s belongs to the $\hat{\Gamma}_r$-orbit of ∞ or r?*

(2) *For two distinct rational numbers $s, s' \in I_1 \cup I_2$, when are the unoriented loops α_s and $\alpha_{s'}$ homotopic in $S^3 - K(r)$?*

The following theorem proved in [11] gives a complete answer to Question 2.2(1).

Theorem 2.3. *The loop α_s is null-homotopic in $S^3 - K(r)$ if and only if s belongs to the $\hat{\Gamma}_r$-orbit of ∞ or r. In other words, if $s \in I_1 \cup I_2$, then α_s is not null-homotopic in $S^3 - K(r)$.*

This has the following application to the study of epimorphisms between 2-bridge link groups (see [11, Section 2] for precise meaning).

Corollary 2.4. *There is an upper-meridian-pair-preserving epimorphism from $G(K(s))$ to $G(K(r))$ if and only if s or $s+1$ belongs to the $\hat{\Gamma}_r$-orbit of r or ∞.*

The following theorem proved in [12, 13, 14] gives a complete answer to Question 2.2(2).

Theorem 2.5. *Suppose that r is a rational number such that $0 < r \leq 1/2$. For distinct $s, s' \in I_1 \cup I_2$, the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$ if and only if one of the following holds.*

(1) *$r = 1/p$, where $p \geq 2$ is an integer, and $s = q_1/p_1$ and $s' = q_2/p_2$ satisfy $q_1 = q_2$ and $q_1/(p_1+p_2) = 1/p$, where (p_i, q_i) is a pair of relatively prime positive integers.*

(2) *$r = 3/8$, namely $K(r)$ is the Whitehead link, and the set $\{s, s'\}$ equals either $\{1/6, 3/10\}$ or $\{3/4, 5/12\}$.*

The proof of Theorem 2.5 reveals the structure of the normalizer of an element of $G(K(r))$ represented by α_s. This enables us to show the following.

Theorem 2.6. *Let r be a rational number such that $0 < r \leq 1/2$. Suppose $K(r)$ is hyperbolic, i.e., $r = q/p$ and $q \not\equiv \pm 1 \pmod{p}$, and let s be a rational number contained in $I_1 \cup I_2$.*

(1) *The loop α_s is peripheral if and only if one of the following holds.*

(i) *$r = 2/5$ and $s = 1/5$ or $s = 3/5$.*

(ii) *$r = n/(2n+1)$ for some integer $n \geq 3$, and $s = (n+1)/(2n+1)$.*

(iii) *$r = 2/(2n+1)$ for some integers $n \geq 3$, and $s = 1/(2n+1)$.*

(2) *The conjugacy class α_s is primitive in $G(K(r))$ with the following exceptions.*

(i) *$r = 2/5$ and $s = 2/7$ or $3/4$. In this case α_s is the third power of some primitive element in $G(K(r))$.*

(ii) *$r = 3/7$ and $s = 2/7$. In this case α_s is the second power of some primitive element in $G(K(r))$.*

(iii) $r = 2/7$ *and* $s = 3/7$. *In this case* α_s *is the second power of some primitive element in* $G(K(r))$.

At the end of this section, we describe a relation of Theorem 2.3 with the question raised by Minsky in [9, Question 5.4]. Let $M = H_+ \cup_S H_-$ be a Heegaard splitting of a 3-manifold M. Let $\Gamma_\pm := MCG(H_\pm)$ be the mapping class group of $H_\pm$, and let $\Gamma^0_\pm$ be the kernel of the map $MCG(H_\pm) \to \mathrm{Out}(\pi_1(H_\pm))$. Identify $\Gamma^0_\pm$ with a subgroup of $MCG(S)$, and consider the subgroup $\langle \Gamma^0_+, \Gamma^0_- \rangle$ of $MCG(S)$. Now let $\Delta_\pm$ be the set of (isotopy classes of) simple loops in S which bound a disk in $H_\pm$. Let Z be the set of essential simple loops in S which are null-homotopic in M. Note that Z contains $\Delta_\pm$ and invariant under $\langle \Gamma^0_+, \Gamma^0_- \rangle$. In particular, the orbit $\langle \Gamma^0_+, \Gamma^0_- \rangle(\Delta_+ \cup \Delta_-)$ is a subset of Z. Then Minsky posed the following question.

Question 2.7. *When is* Z *equal to the orbit* $\langle \Gamma^0_+, \Gamma^0_- \rangle(\Delta_+ \cup \Delta_-)$*?*

The above question makes sense not only for Heegaard splittings but also bridge decompositions of knots and links. In particular, for 2-bridge links, the groups Γ_∞ and Γ_r in our setting correspond to the groups Γ^0_+ and Γ^0_-, and hence the group $\hat{\Gamma}_r$ corresponds to the group $\langle \Gamma^0_+, \Gamma^0_- \rangle$. To make this precise, recall the bridge decomposition $(S^3, K(r)) = (B^3, t(\infty)) \cup (B^3, t(r))$, and let $\tilde{\Gamma}_+$ (resp. $\tilde{\Gamma}_-$) be the mapping class group of the pair $(B^3, t(\infty))$ (resp. $(B^3, t(r))$), and let $\tilde{\Gamma}^0_\pm$ be the kernel of the natural map $\tilde{\Gamma}_+ \to \mathrm{Out}(\pi_1(B^3 - t(\infty)))$ (resp. $\tilde{\Gamma}_- \to \mathrm{Out}(\pi_1(B^3 - t(r)))$). Identify $\tilde{\Gamma}^0_\pm$ with a subgroup of the mapping class group $MCG(\boldsymbol{S})$ of the 4-times punctured sphere $\boldsymbol{S}$. Recall that the Farey tessellation $\mathcal{D}$ is identified with the curve complex of $\boldsymbol{S}$ and there is a natural epimorphism from $MCG(\boldsymbol{S})$ to the automorphism group $Aut(\mathcal{D})$ of $\mathcal{D}$, whose kernel is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Then the groups Γ_∞ and Γ_r, respectively, are identified with the images of $\tilde{\Gamma}^0_+$ and $\tilde{\Gamma}^0_-$ by this epimorphism. Moreover, the sets $\{\alpha_\infty\}$ and $\{\alpha_r\}$, respectively, correspond to the sets Δ_+ and Δ_-. Theorem 2.3 says that the set Z of simple loops in $\boldsymbol{S}$ which are null-homotopic in $S^3 - K(r)$ is equal to the orbit $\langle \Gamma_\infty, \Gamma_r \rangle(\Delta_+ \cup \Delta_-)$. Thus Theorem 2.3 may be regarded as an answer to the special variation of Question 2.7.

3. Application to character varieties

In this section and the next section, we assume $r = q/p$, where p and q are relatively prime positive integers such that $q \not\equiv \pm 1 \pmod p$. This is equivalent to the condition that $K(r)$ is hyperbolic, namely the link complement $S^3 - K(r)$ admits a complete hyperbolic structure of finite volume. Let ρ_r be the

$\mathrm{PSL}(2,\mathbb{C})$-representation of $\pi_1(\boldsymbol{S})$ obtained as the composition

$$\pi_1(\boldsymbol{S}) \to \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle \cong \pi_1(S^3 - K(r)) \to \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2,\mathbb{C}),$$

where the last homomorphism is the holonomy representation associated with the complete hyperbolic structure.

Now, let $\boldsymbol{T}$ be the once-punctured torus obtained as the quotient $(\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$, and let $\boldsymbol{O}$ be the orbifold $(\mathbb{R}^2 - \mathbb{Z}^2)/\hat{H}$ where $\hat{H}$ is the group generated by π-rotations around the points in $(\frac{1}{2}\mathbb{Z})^2$. Note that $\boldsymbol{O}$ is the orbifold with underlying space a once-punctured sphere and with three cone points of cone angle π. The surfaces $\boldsymbol{T}$ and $\boldsymbol{S}$, respectively, are $\mathbb{Z}/2\mathbb{Z}$-covering and $(\mathbb{Z}/2\mathbb{Z})^2$-covering of $\boldsymbol{O}$, and hence their fundamental groups are identified with subgroups of the orbifold fundamental group $\pi_1(\boldsymbol{O})$ of indices 2 and 4, respectively. The $\mathrm{PSL}(2,\mathbb{C})$-representation ρ_r of $\pi_1(\boldsymbol{S})$ extends, in a unique way, to that of $\pi_1(\boldsymbol{O})$ (see [4, Proposition 2.2]), and so we obtain, in a unique way, a $\mathrm{PSL}(2,\mathbb{C})$-representation of $\pi_1(\boldsymbol{T})$ by restriction. We continue to denote it by ρ_r. Note that $\rho_r : \pi_1(\boldsymbol{T}) \to \mathrm{PSL}(2,\mathbb{C})$ is *type-preserving*, i.e., it satisfies the following conditions.

(1) ρ_r is irreducible, i.e., its image does not have a common fixed point on $\partial\mathbb{H}^3$.

(2) ρ_r maps a peripheral element of $\pi_1(\boldsymbol{T})$ to a parabolic transformation.

By extending the concept of a geometrically infinite end of a Kleinian group, Bowditch [7] introduced the notion of the end invariants of a type-preserving $\mathrm{PSL}(2,\mathbb{C})$-representation of $\pi_1(\boldsymbol{T})$. Tan, Wong and Zhang [30] (cf. [26]) extended this notion (with slight modification) to an arbitrary $\mathrm{PSL}(2,\mathbb{C})$-representation of $\pi_1(\boldsymbol{T})$. (To be precise, [30] treats $\mathrm{SL}(2,\mathbb{C})$-representations. However, the arguments work for $\mathrm{PSL}(2,\mathbb{C})$-representations.)

To recall the notion of end invariants, let $\mathcal{C}$ be the set of free homotopy classes of essential simple loops on $\boldsymbol{T}$. Then $\mathcal{C}$ is identified with $\hat{\mathbb{Q}}$, the vertex set of the Farey tessellation $\mathcal{D}$ by the following rule. For each $s \in \hat{\mathbb{Q}}$, let β_s be the essential simple loop on $\boldsymbol{T}$ obtained as the image of a line of slope s in $\mathbb{R}^2 - \mathbb{Z}^2$. Then the correspondence $s \mapsto \beta_s$ gives the desired identification $\hat{\mathbb{Q}} \cong \mathcal{C}$. The projective lamination space $\mathcal{PL}$ is then identified with $\hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and contains $\mathcal{C}$ as the dense subset of rational points.

Definition 3.1. Let ρ be a $\mathrm{PSL}(2,\mathbb{C})$-representation of $\pi_1(\boldsymbol{T})$.

(1) An element $X \in \mathcal{PL}$ is an *end invariant* of ρ if there exists a sequence of distinct elements $X_n \in \mathcal{C}$ such that $X_n \to X$ and such that $\{|\mathrm{tr}\rho(X_n)|\}_n$ is bounded from above.

(2) $\mathcal{E}(\rho)$ denotes the set of end invariants of ρ.

In the above definition, it should be noted that $|\mathrm{tr}\rho(X_n)|$ is well-defined though $\mathrm{tr}\rho(X_n)$ is defined only up to sign. Note also that the condition that $\{|\mathrm{tr}\rho(X_n)|\}_n$ is bounded from above is equivalent to the condition that the hyperbolic translation lengths of the isometries $\rho(X_n)$ of $\mathbb{H}^3$ are bounded from above.

Tan, Wong and Zhang [26, 30] showed that $\mathcal{E}(\rho)$ is a closed subset of $\mathcal{PL}$ and proved various interesting properties of $\mathcal{E}(\rho)$, including a characterization of those representations ρ with $\mathcal{E}(\rho) = \emptyset$ or $\mathcal{PL}$, generalizing a result of Bowditch [7]. They also proposed an interesting conjecture [30, Conjecture 1.8] concerning possible homeomorphism types of $\mathcal{E}(\rho)$. The following is a modified version of the conjecture of which Tan [25] informed the authors.

Conjecture 3.2. Suppose $\mathcal{E}(\rho)$ has at least two accumulation points. Then either $\mathcal{E}(\rho) = \mathcal{PL}$ or a Cantor set of $\mathcal{PL}$.

They constructed a family of representations ρ which have Cantor sets as $\mathcal{E}(\rho)$, and proved the following supporting evidence to the conjecture.

Theorem 3.3. *Let $\rho : \pi_1(\boldsymbol{T}) \to \mathrm{SL}(2,\mathbb{C})$ be* discrete *in the sense that the set $\{\mathrm{tr}(\rho(X)) \mid X \in \mathcal{C}\}$ is discrete in $\mathbb{C}$. Then if $\mathcal{E}(\rho)$ has at least three elements, then $\mathcal{E}(\rho)$ is either a Cantor set of $\mathcal{PL}$ or all of $\mathcal{PL}$.*

The above theorem implies that the end invariants $\mathcal{E}(\rho_r)$ of the representation ρ_r induced by the holonomy representation of a hyperbolic 2-bridge link $K(r)$ is a Cantor set. But it does not give us the exact description of $\mathcal{E}(\rho_r)$. By using the main results stated in Section 2, we can explicitly determine the end invariants $\mathcal{E}(\rho_r)$. To state the theorem, recall that the *limit set* $\Lambda(\hat{\Gamma}_r)$ of the group $\hat{\Gamma}_r$ is the set of accumulation points in the closure of $\mathbb{H}^2$ of the $\hat{\Gamma}_r$-orbit of a point in $\mathbb{H}^2$.

Theorem 3.4. *For a hyperbolic 2-bridge link $K(r)$, the set $\mathcal{E}(\rho_r)$ is equal to the limit set $\Lambda(\hat{\Gamma}_r)$ of the group $\hat{\Gamma}_r$.*

We would like to propose the following conjecture.

Conjecture 3.5. Let $\rho : \pi_1(\boldsymbol{T}) \to \mathrm{PSL}(2,\mathbb{C})$ be a type-preserving representation such that $\mathcal{E}(\rho) = \Lambda(\hat{\Gamma}_r)$. Then ρ is conjugate to the representation ρ_r.

4. Application to McShane's identity

In his Ph.D. thesis [17], McShane proved the following surprising theorem.

Theorem 4.1. *Let $\rho : \pi_1(\boldsymbol{T}) \to \mathrm{PSL}(2,\mathbb{R})$ be a type-preserving fuchsian representation. Then*

$$2\sum_{s\in\hat{\mathbb{Q}}} \frac{1}{1+e^{l_\rho(\beta_s)}} = \frac{1}{2}$$

In the above identity, $l_\rho(\beta_s)$ denotes the translation length of the orientation-preserving isometry $\rho(\beta_s)$ of the hyperbolic plane. This identity has been generalized to cusped hyperbolic surfaces by McShane himself [18], to hyperbolic surfaces with cusps and geodesic boundary by Mirzakhani [19], and to hyperbolic surfaces with cusps, geodesic boundary and conical singularities by Tan, Wong and Zhang [27]. A wonderful application to the Weil-Petersson volume of the moduli spaces of bordered hyperbolic surface was found by Mirzakhani [19]. Bowditch [7] (cf. [6]) showed that the identity in Theorem 4.1 is also valid for all quasifuchsian representations of $\pi_1(\boldsymbol{T})$, where $l_\rho(\beta_s)$ is regarded as the complex translation length of the orientation-preserving isometry $\rho(\beta_s)$ of the hyperbolic 3-space. Moreover, he gave a nice variation of the identity for hyperbolic once-punctured torus bundles, which describes the cusp shape in terms of the complex translation lengths of essential simple loops on the fiber torus [8]. Other 3-dimensional variations have been obtained by [2, 3, 26, 27, 28, 29, 30, 31].

As an application of the main results stated in Section 2, we can obtain yet another 3-dimensional variation of McShane's identity, which describes the cusp shape of a hyperbolic 2-bridge link in terms of the complex translation lengths of essential simple loops on the bridge sphere. This proves a conjecture proposed by the first author in [23].

To describe the result, note that each cusp of the hyperbolic manifold $S^3 - K(r)$ carries a Euclidean structure, well-defined up to similarity, and hence it is identified with the quotient of $\mathbb{C}$ (with the natural Euclidean metric) by the lattice $\mathbb{Z}\oplus\mathbb{Z}\lambda$, generated by the translations $[\zeta \mapsto \zeta+1]$ and $[\zeta\mapsto\zeta+\lambda]$ corresponding to the meridian and (suitably chosen) longitude respectively. This λ does not depend on the choice of the cusp, because when $K(r)$ is a two-component link there is an isometry of $S^3 - K(r)$ interchanging the two cusps. We call λ the *modulus* of the cusp and denote it by $\lambda(K(r))$.

Theorem 4.2. *For a hyperbolic 2-bridge link $K(r)$ with $r = q/p$, the following identity holds:*

$$2\sum_{s\in \mathrm{int} I_1} \frac{1}{1+e^{l_{\rho_r}(\beta_s)}} + 2\sum_{s\in \mathrm{int} I_2} \frac{1}{1+e^{l_{\rho_r}(\beta_s)}} + \sum_{s\in \partial I_1\cup\partial I_2} \frac{1}{1+e^{l_{\rho_r}(\beta_s)}} = -1.$$

Further the modulus $\lambda(K(r))$ of the cusp torus of the cusped hyperbolic manifold $S^3 - K(r)$ with respect to a suitable choice of a longitude is given by the following formula:

$$\lambda(K(r)) = \begin{cases} 8\sum_{s\in \mathrm{int} I_1} \frac{1}{1+e^{l_{\rho_r}(\beta_s)}} + 4\sum_{s\in\partial I_1} \frac{1}{1+e^{l_{\rho_r}(\beta_s)}} & \text{if } p \text{ is odd,} \\ 4\sum_{s\in \mathrm{int} I_1} \frac{1}{1+e^{l_{\rho_r}(\beta_s)}} + 2\sum_{s\in\partial I_1} \frac{1}{1+e^{l_{\rho_r}(\beta_s)}} & \text{if } p \text{ is even.} \end{cases}$$

The main results stated in Section 2 are used to establish the absolute convergence of the infinite series.

5. Presentations of 2-bridge link groups

In the remainder of this note, p and q denote relatively prime positive integers such that $1 \le q \le p$ and $r = q/p$. Theorems 2.3 and 2.5 are proved by applying the small cancellation theory to a two-generator and one-relator presentation of the link group $G(K(r))$. To recall the presentation, let a and b, respectively, be the elements of $\pi_1(B^3 - t(\infty), x_0)$ represented by the oriented loops μ_1 and μ_2 based on x_0 as illustrated in Figure 3. Then $\pi_1(B^3 - t(\infty), x_0)$ is identified with the free group $F(a, b)$. Note that μ_i intersects the disk, δ_i, in B^3 bounded by a component of $t(\infty)$ and the essential arc, γ_i, on $\partial(B^3, t(\infty)) = (\boldsymbol{S}^2, \boldsymbol{P})$ of slope $1/0$, in Figure 3.

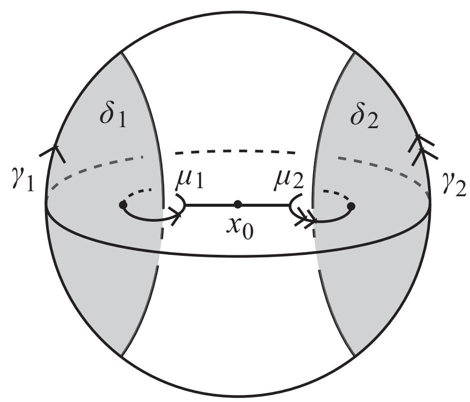

Figure 3. $\pi_1(B^3 - t(\infty), x_0) = F(a, b)$, where a and b are represented by μ_1 and μ_2, respectively.

To obtain an element, u_r, of $F(a, b)$ represented by the simple loop α_r (with a suitable choice of an orientation and a path joining α_r to the base point x_0), note that the inverse image of γ_1 (resp. γ_2) in $\mathbb{R}^2 - \mathbb{Z}^2$ is the union of the single arrowed (resp. double arrowed) vertical edges in Figure 4. Let

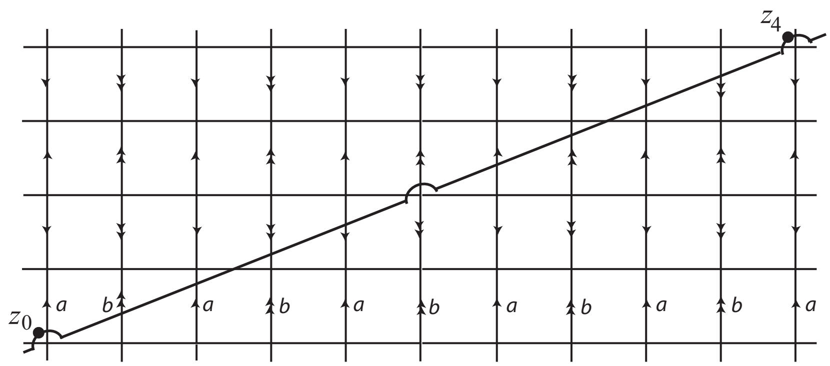

FIGURE 4. The line of slope $2/5$ gives $\hat{u}_{2/5} = bab^{-1}a^{-1}$ and $u_{2/5} = a\hat{u}_{2/5}b\hat{u}_{2/5}^{-1} = abab^{-1}a^{-1}baba^{-1}b^{-1}$. Since the inverse image of γ_1 (resp. γ_2) in $\mathbb{R}^2 - \mathbb{Z}^2$ is the union of the single arrowed (resp. double arrowed) vertical edges, a positive intersection with a single arrowed (resp. double arrowed) edge corresponds to a (resp. b).

$L(r)$ be the line in $\mathbb{R}^2$ of slope r passing through the origin, and let $L^+(r)$ be the line in $\mathbb{R}^2 - \mathbb{Z}^2$ obtained by slightly modifying $L(r)$ near each of the lattice points in $L(r)$ so that $L^+(r)$ takes an upper circuitous route around it, as illustrated in Figure 4. Pick a base point z_0 from the intersection of $L^+(r)$ with the second quadrant, and consider the sub-line-segment of $L^+(r)$ bounded by z_0 and $z_4 := z_0 + (2p, 2q)$. Then the image of the sub-line-segment in $\boldsymbol{S}$ is homotopic to the loop α_s. Let u_r be the word in $\{a, b\}$ obtained by reading the intersection of the line-segment with the vertical lattice lines (= the inverse images of γ_1 and γ_2) as in Figure 4. Then $u_r \in F(a, b) \cong \pi_1(B^3 - t(\infty))$ is represented by the simple loop α_r, and we obtain the following two-generator one-relator presentation.

$$\begin{aligned} G(K(r)) = \pi_1(S^3 - K(r)) &\cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r \rangle\rangle \\ &\cong F(a, b)/\langle\langle u_r \rangle\rangle \cong \langle a, b \,|\, u_r \rangle. \end{aligned}$$

To describe the explicit formula for u_r, set $\epsilon_i = (-1)^{\lfloor iq/p \rfloor}$ where $\lfloor x \rfloor$ is the greatest integer not exceeding x. Then we have the following (cf. [22, Proposition 1]). Let

$$\epsilon_i = (-1)^{\lfloor iq/p \rfloor},$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x.

(1) If p is odd, then

$$u_{q/p} = a\hat{u}_{q/p} b^{(-1)^q} \hat{u}_{q/p}^{-1},$$

where $\hat{u}_{q/p} = b^{\epsilon_1} a^{\epsilon_2} \cdots b^{\epsilon_{p-2}} a^{\epsilon_{p-1}}$.

(2) If p is even, then

$$u_{q/p} = a\hat{u}_{q/p} a^{-1} \hat{u}_{q/p}^{-1},$$

where $\hat{u}_{q/p} = b^{\epsilon_1} a^{\epsilon_2} \cdots a^{\epsilon_{p-2}} b^{\epsilon_{p-1}}$.

In the above formula, $\hat{u}_{q/p}$ is obtained from the open interval of $L(r)$ bounded by $(0,0)$ and (p,q).

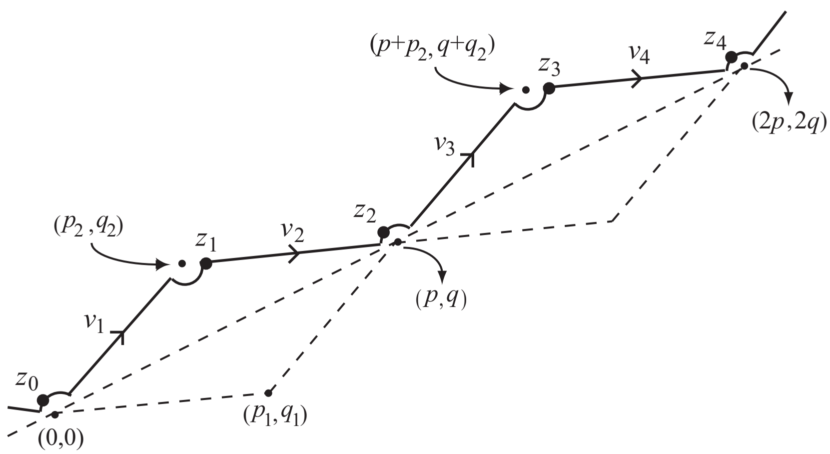

FIGURE 5. The decomposition of the relator $u_r = v_1 v_2 v_3 v_4$

We now describe a natural decomposition of the word u_r, which plays a key role in the proof of the main results. Let $r_i = q_i/p_i$ $(i = 1, 2)$ be the rational number introduced in Section 2. Then $(p, q) = (p_1 + p_2, q_1 + q_2)$ and the parallelogram in $\mathbb{R}^2$ spanned by $(0,0)$, (p_1, q_1), (p_2, q_2) and (p, q) does not contain lattice points in its interior. Consider the infinite broken line, $L_b(r)$, obtained by joining the lattice points

$$\ldots, (0,0), (p_2, q_2), (p, q), (p + p_2, q + q_2), (2p, 2q), \ldots$$

which is invariant by the translation $(x, y) \mapsto (x + p, y + q)$. Let $L_b^+(r)$ be the topological line obtained by slightly modifying $L_b(r)$ near each of the lattice points in $L_b(r)$ so that $L_b^+(r)$ takes an upper or lower circuitous route around it according as the lattice point is of the form $d(p, q)$ or $d(p, q) + (p_2, q_2)$ for some $d \in \mathbb{Z}$, as illustrated in Figure 5. We may assume the base points z_0 and z_4 in $L^+(r)$ also lie in $L_b^+(r)$. Then the sub-arcs of $L^+(r)$ and $L_b^+(r)$

bounded by z_0 and z_4 are homotopic in $\mathbb{R}^2 - \mathbb{Z}^2$ by a homotopy fixing the end points. Moreover, the word u_r is also obtained by reading the intersection of the sub-path of $L_b^+(r)$ with the vertical lattice lines. Pick a point $z_1 \in L_b^+(r)$ whose x-coordinate is $p_2 +$ (small positive number), and set $z_2 := z_0 + (p, q)$ and $z_3 := z_1 + (p, q)$. Let $L_{b,i}^+(r)$ be the sub-path of $L_b^+(r)$ bounded by z_{i-1} and z_i $(i = 1, 2, 3, 4)$, and consider the subword, v_i, of u_r corresponding to $L_{b,i}^+(r)$. Then we have the decomposition

$$u_r = v_1 v_2 v_3 v_4,$$

where the lengths of the subwords v_i are given by $|v_1| = |v_3| = p_2 + 1$ and $|v_2| = |v_4| = p_1 - 1$. This decomposition plays a key role in the following section.

6. Sequences associated with the simple loop α_r

We begin with the following observation.

(1) The word u_r is *reduced*, i.e., it does not contain xx^{-1} or $x^{-1}x$ for any $x \in \{a, b\}$. It is also *cyclically reduced*, i.e., all its cyclic permutations are reduced.

(2) The word u_r is *alternating*, i.e., $a^{\pm 1}$ and $b^{\pm 1}$ appear in u_r alternately, to be precise, neither $a^{\pm 2}$ nor $b^{\pm 2}$ appears in u_r. It is also *cyclically alternating*, i.e., all its cyclic permutations are alternating.

This observation implies that the word u_r is determined by the S-sequence defined below and the initial letter (with exponent).

Definition 6.1. (1) Let w be a nonempty reduced word in $\{a, b\}$. Decompose w into

$$w \equiv w_1 w_2 \cdots w_t,$$

where, for each $i = 1, \dots, t-1$, all letters in w_i have positive (resp. negative) exponents, and all letters in w_{i+1} have negative (resp. positive) exponents. (Here the symbol $\equiv$ means that the two words are not only equal as elements of the free group but also visibly equal, i.e., equal without cancellation.) Then the sequence of positive integers $S(w) := (|w_1|, |w_2|, \dots, |w_t|)$ is called the *S-sequence of v*.

(2) Let (w) be a nonempty reduced cyclic word in $\{a, b\}$ represented by a word w. Decompose (w) into

$$(w) \equiv (w_1 w_2 \cdots w_t),$$

where all letters in w_i have positive (resp. negative) exponents, and all letters in w_{i+1} have negative (resp. positive) exponents (taking subindices modulo t).

Then the *cyclic* sequence of positive integers $CS(w) := ((|w_1|, |w_2|, \dots, |w_t|))$ is called the *cyclic S-sequence of* (w). Here the double parentheses denote that the sequence is considered modulo cyclic permutations.

In the above definition, by a *cyclic word*, we mean the set of all cyclic permutations of a cyclically reduced word. By (v), we denote the cyclic word associated with a cyclically reduced word v.

Definition 6.2. For a rational number r with $0 < r \le 1$, let u_r be the word in $\{a, b\}$ defined in Section 5. Then the symbol $S(r)$ (resp. $CS(r)$) denotes the S-sequence $S(u_r)$ of u_r (resp. cyclic S-sequence $CS(u_r)$ of (u_r)), which is called the *S-sequence of slope* r (resp. the *cyclic S-sequence of slope* r).

We can easily observe the following.

$$S(r) = S(u_r) = (S(v_1), S(v_2), S(v_3), S(v_4)),$$
$$CS(r) = CS(u_r) = ((S(v_1), S(v_2), S(v_3), S(v_4))),$$

where $u_r = v_1 v_2 v_3 v_4$ is the natural decomposition of u_r obtained at the end of the last section. It is also not difficult to observe $S(v_1) = S(v_3)$ and $S(v_2) = S(v_4)$. By setting $S_1 := S(v_1) = S(v_3)$ and $S_2 := S(v_2) = S(v_4)$, we have the following key propositions.

Proposition 6.3. *The decomposition $S(r) = (S_1, S_2, S_1, S_2)$ satisfies the following.*

(1) *Each S_i is symmetric, i.e., the sequence obtained from S_i by reversing the order is equal to S_i. (Here, S_1 is empty if $k = 1$.)*
(2) *Each S_i occurs only twice in the cyclic sequence $CS(r)$.*
(3) *Set $m := \lfloor q/p \rfloor$. Then $S(r)$ consists of only m and $m+1$, and S_1 begins and ends with $m+1$, whereas S_2 begins and ends with m.*

Proposition 6.4. *Let $S(r) = (S_1, S_2, S_1, S_2)$ be as in Proposition 6.3. For a rational number s with $0 < s \le 1$, suppose that the cyclic S-sequence $CS(s)$ contains both S_1 and S_2 as a subsequence. Then $s \notin I_1 \cup I_2$.*

7. Small cancellation conditions for 2-bridge link groups

A subset R of the free group $F(a, b)$ is called *symmetrized*, if all elements of R are cyclically reduced and, for each $w \in R$, all cyclic permutations of w and w^{-1} also belong to R.

Definition 7.1. Suppose that R is a symmetrized subset of $F(a, b)$. A nonempty word v is called a *piece* if there exist distinct $w_1, w_2 \in R$ such that $w_1 \equiv vc_1$

and $w_2 \equiv vc_2$. Small cancellation conditions $C(p)$ and $T(q)$, where p and q are integers such that $p \geq 2$ and $q \geq 3$, are defined as follows (see [16]).

(1) Condition $C(p)$: If $w \in R$ is a product of n pieces, then $n \geq p$.
(2) Condition $T(q)$: For $w_1, \dots, w_n \in R$ with no successive elements w_i, w_{i+1} an inverse pair (i mod n), if $n < q$, then at least one of the products $w_1 w_2, \dots, w_{n-1}w_n$, $w_n w_1$ is freely reduced without cancellation.

The following proposition enables us to apply small cancellation theory to the group presentation $\langle a, b \,|\, u_r \rangle$ of $G(K(r))$.

Proposition 7.2. *Let r be a rational number such that $0 < r < 1$, and let R be the symmetrized subset of $F(a,b)$ generated by the single relator u_r of the group presentation $G(K(r)) = \langle a, b \,|\, u_r \rangle$. Then R satisfies $C(4)$ and $T(4)$.*

This proposition follows from the following characterization of pieces, which in turn is proved by using Proposition 6.3.

Proposition 7.3. (1) *A subword w of the cyclic word $(u_r^{\pm 1})$ is a piece if and only if $S(w)$ does not contain S_1 as a subsequence and does not contain S_2 in its interior, i.e., $S(w)$ does not contain a subsequence (ℓ_1, S_2, ℓ_2) for some $\ell_1, \ell_2 \in \mathbb{Z}_+$.*

(2) *For a subword w of the cyclic word $(u_r^{\pm 1})$, w is not a product of two pieces if and only if $S(w)$ either contains (S_1, S_2) as a proper initial subsequence or contains (S_2, S_1) as a proper terminal subsequence.*

8. Outline of the proof of Theorem 2.3

Let R be the symmetrized subset of $F(a,b)$ generated by the single relator u_r of the group presentation $G(K(r)) = \langle a, b \mid u_r \rangle$. Suppose on the contrary that α_s is null-homotopic in $S^3 - K(r)$, i.e., $u_s = 1$ in $G(K(r))$, for some $s \in I_1 \cup I_2$. Then there is a *van Kampen diagram* M over $G(K(r)) = \langle a, b \,|\, R \rangle$ such that the boundary label is u_s. Here M is a simply connected 2-dimensional complex embedded in $\mathbb{R}^2$, together with a function ϕ assigning to each oriented edge e of M, as a *label*, a reduced word $\phi(e)$ in $\{a, b\}$ such that the following hold.

(1) If e is an oriented edge of M and e^{-1} is the oppositely oriented edge, then $\phi(e^{-1}) = \phi(e)^{-1}$.
(2) For any boundary cycle δ of any face of M, $\phi(\delta)$ is a cyclically reduced word representing an element of R. (If $\alpha = e_1, \dots, e_n$ is a path in M, we define $\phi(\alpha) \equiv \phi(e_1) \cdots \phi(e_n)$.)

We may assume M is *reduced*, namely it satisfies the following condition: Let D_1 and D_2 be faces (not necessarily distinct) of M with an edge $e \subseteq \partial D_1 \cap \partial D_2$,

and let $e\delta_1$ and $\delta_2 e^{-1}$ be boundary cycles of D_1 and D_2, respectively. Set $\phi(\delta_1) = f_1$ and $\phi(\delta_2) = f_2$. Then we have $f_2 \neq f_1^{-1}$. Moreover, we may assume the following conditions:

(1) $d_M(v) \geq 3$ for every vertex $v \in M - \partial M$.
(2) For every edge e of ∂M, the label $\phi(e)$ is a piece.
(3) For a path $e_1, \dots, e_n$ in ∂M of length $n \geq 2$ such that the vertex $e_i \cap e_{i+1}$ has degree 2 for $i = 1, 2, \dots, n-1$, $\phi(e_1)\phi(e_2)\cdots\phi(e_n)$ cannot be expressed as a product of less than n pieces.

Since R satisfies the conditions $C(4)$ and $T(4)$ by Proposition 7.2, M is a $[4,4]$-map, i.e.,

(1) $d_M(v) \geq 4$ for every vertex $v \in M - \partial M$;
(2) $d_M(D) \geq 4$ for every face $D \in M$.

Here, $d_M(v)$, the *degree of* v, denotes the number of oriented edges in M having v as initial vertex, and $d_M(D)$, the *degree of* D, denotes the number of oriented edges in a boundary cycle of D.

Now, for simplicity, assume that M is homeomorphic to a disk. (In general, we need to consider an extremal disk of M.) Then by the Curvature Formula of Lyndon and Schupp (see [16, Corollary V.3.4]), we have

$$\sum_{v \in \partial M} (3 - d_M(v)) \geq 4.$$

By using this formula, we see that there are three edges e_1, e_2 and e_3 in ∂M such that $e_1 \cap e_2 = \{v_1\}$ and $e_2 \cap e_3 = \{v_2\}$, where $d_M(v_i) = 2$ for each $i = 1, 2$. Since $\phi(e_1)\phi(e_2)\phi(e_3)$ is not expressed as a product of two pieces, we see by Proposition 7.3 that the boundary label of M contains a subword, w, with $S(w) = (S_1, S_2, \ell)$ or (ℓ, S_2, S_1). This in turn implies that the S-sequence of the boundary label contains both S_1 and S_2 as subsequences. Hence, by Proposition 6.4, we have $s \notin I_1 \cup I_2$, a contradiction.

9. Outline of the proof of Theorem 2.5

Suppose, for two distinct $s, s' \in I_1 \cup I_2$, the unoriented loops α_s and $\alpha_{s'}$ are homotopic in $S^3 - K(r)$. Then there is a reduced annular R-diagram M such that u_s is an outer boundary label and $u_{s'}^{\pm 1}$ is an inner boundary label of M. Again we can see that M is a $[4,4]$-map and hence we have the following curvature formula:

$$\sum_{v \in \partial M} (3 - d_M(v)) \geq 0.$$

By using this formula, we obtain the following very strong structure theorem for M, which plays key roles throughout the series of papers [12, 13, 14].

Theorem 9.1. *Figure 6(a) illustrates the only possible type of the outer boundary layer of M, while Figure 6(b) illustrates the only possible type of whole M. (The number of faces per layer and the number of layers are variable.)*

In the above theorem, the *outer boundary layer* of the annular map M is the submap of M consisting of all faces D such that the intersection of ∂D with the outer boundary of M contains an edge, together with the edges and vertices contained in ∂D.

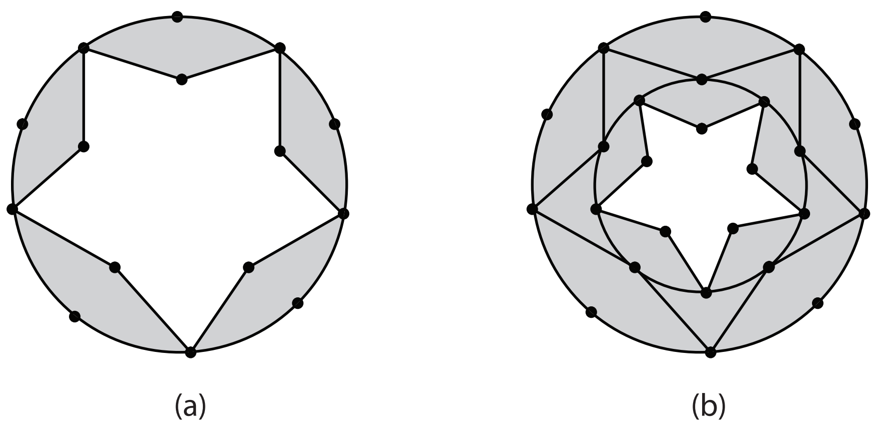

FIGURE 6.

The first paper [12] of the series devoted to proof of Theorem 2.5 treats the case when the 2-bridge link is a $(2, p)$-torus link, the second paper [13] treats the case of 2-bridge links of slope $n/(2n+1)$ and $(n+1)/(3n+2)$, where $n \geq 2$ is an arbitrary integer, and the third paper [14] treats the general case. The two families treated in the second paper play special roles in the project in the sense that the treatment of these links form a base step of an inductive proof of the theorem for general 2-bridge links. We note that the figure-eight knot is both a 2-bridge link of slope $n/(2n+1)$ with $n = 2$ and a 2-bridge link of slope $(n+1)/(3n+2)$ with $n = 1$. Surprisingly, the treatment of the figure-eight knot, the simplest hyperbolic knot, is the most complicated. This reminds us of the phenomenon in the theory of exceptional Dehn filling that the figure-eight knot attains the maximal number of exceptional Dehn fillings.

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Department of Mathematics, Pusan National University, San-30 Jangjeon-Dong, Geumjung-Gu, Pusan, 609-735, Republic of Korea
E-mail address: donghi@pusan.ac.kr

Department of Mathematics, Graduate School of Science, Hiroshima University, Higashi-Hiroshima, 739-8526, Japan
E-mail address: sakuma@math.sci.hiroshima-u.ac.jp